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Existence of Solutions for some Nonlinear Elliptic Problem with Measure Data in Musielak-Orlicz-Sobolev Spaces

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Abstract

This paper establishes the existence of solutions for a class of nonlinear elliptic problems of the form:

$$A(u) + g(x, u, \nabla u) + div\, \sigma(u) = \mu,$$

where g and σ are lower-order terms satisfying specific growth and sign conditions. The term μ represents a bounded nonnegative Radon measure on a domain $\Omega \subset R^N$, and $\sigma : R \to R^N$ is a continuous function. The analysis is conducted within the framework of Musielak-Orlicz-Sobolev spaces.

Keywords: Musielak-Orlicz-Sobolev spaces, Nonlinear elliptic problems, Measure data, Weak solutions.

إيجاد الحلول لبعض مسائل القطع الناقص غير الخطية مع بيانات القياس في فضاءات سبوليف-أورلكس-ميزيلاك

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الملخص: تهدف هذه الورقة البحثية إلى إثبات وجود حلول لفئة من مسائل القطع الناقص غير الخطية من الشكل:

$$A(u) + g(x, u, \nabla u) + \operatorname{div} \sigma(u) = \mu$$

حيث تمثل g و σ حدوداً ذات رتب دنيا تستوفي شروط نمو وإشارة محددة، وμ هو قياس رادون محدود غير سالب على نطاق $\Omega \subset R^N$، و $\sigma : R \to R^N$ دالة مستمرة. يتم التحليل في إطار فضاءات سبوليف-أورلكس-ميزيلاك.

الكلمات المفتاحية: فضاءات سبوليف-أورلكس-ميزيلاك مسائل القطع الناقص غير الخطية، بيانات القياس، الحلول الضعيفة.

1 Introduction

Partial differential equations (PDEs) represent a cornerstone in modeling natural phenomena, and their study has engaged mathematicians since the 17th century through the works of Newton, Euler, D'Alembert, Lagrange, and Laplace, among others. These models arise in diverse fields such as electromagnetic theory, quantum mechanics, general relativity, and fluid dynamics. A central focus in the analysis of PDEs is the investigation of qualitative properties of solutions, including existence, uniqueness, regularity, and stability. Furthermore, approximate solutions are often constructed, for instance, by sequences of solutions to simpler, regularized problems.

When addressing a PDE, a fundamental step is to formulate it rigorously within appropriate functional spaces and to establish the existence and uniqueness of solutions. This process encounters various challenges depending on the type of equation (e.g., elliptic, parabolic, hyperbolic, or degenerate).

In this work, we consider the following nonlinear elliptic boundary value problem:

$$\begin{cases} A(u) + g(x,u,\nabla u) + div\,\sigma(u) = \mu & in\Omega, \\ u = 0 & on\partial\Omega, \end{cases} \quad 1.1$$

where $A(u) = -div\,a(x,u,\nabla u)$ is a Leray-Lions type operator.

Significant contributions have been made to problems with measure data. For instance:

- Boccardo and Orsina (2000) studied problems with L^1 data
- Akdim, Benkirane, and Rhoudaf (2008) analyzed degenerate problems under general growth conditions
- Subsequent works by Azroul et al. extended these results to different types of data within various functional settings

The analysis is conducted in the context of Musielak-Orlicz-Sobolev spaces, which provide a flexible framework for handling non-standard growth conditions. Previous research in these spaces has addressed problems similar to (1.1), often under a sign condition on the nonlinearity g (e.g., [1], [2], [3], [7]).

The primary objectives and contributions of this paper are twofold. First, we prove the existence of solutions for problem (1.1) without imposing a sign condition on the nonlinearity g. Second, we demonstrate that the solutions belong to the Musielak-Sobolev space $W_0^1L_\sigma(\Omega)$, where σ belongs to a specific class of Musielak-Orlicz functions A_φ (see Definition 3.1).

The paper is structured as follows. Section 2 reviews essential preliminaries on Musielak-Orlicz-Sobolev spaces. Section 3 presents key technical lemmas. The main assumptions are detailed in Section 4. Finally, Section 5 is devoted to stating and proving the principal existence result (Theorem 5.1).

2 Preliminaries

This section introduces fundamental concepts and established results concerning Musielak-Orlicz-Sobolev spaces.

2.1 Musielak-Orlicz Function

Let Ω be an open subset of R^N $(N \geq 2)$, and let $\varphi(x,t)$ be a real-valued function defined in ΩR^+ satisfying:

(a) $\varphi(x,\cdot)$ is an N-function: convex, nondecreasing, continuous, with $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$, and:

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0, \quad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty.$$

(b) $\varphi(\cdot,t)$ is a measurable function.

A function satisfying conditions (a) and (b) is termed a Musielak-Orlicz function.

For a Musielak-Orlicz function $\varphi(x,t)$, we set $\varphi_x(t) = \varphi(x,t)$ and denote by $\varphi_x^{-1}(t)$ the reciprocal function with respect to t of $\varphi_x(t)$, satisfying:

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

For any two Musielak-Orlicz functions $\varphi(x,t)$ and $\gamma(x,t)$, we introduce the following ordering:
(c) If there exist positive constants c and T such that for almost everywhere $x \in \Omega$:

$$\gamma(x,t) \leq \varphi(x,ct) \quad for \quad t \geq T,$$

we write $\gamma \prec\prec \varphi$, and say that φ dominates γ globally if $T = 0$, and near infinity if $T > 0$.
(d) For every positive constant c and almost everywhere $x \in \Omega$, if:

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0 \quad or \quad \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0.$$

Remark 2.1 ([15]) *If $\gamma \prec\prec \varphi$ near infinity, then for every $\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$:*

$$\gamma(x,t) \leq k(\varepsilon)\varphi(x,\varepsilon t) \quad \forall t \geq 0.$$

Remark 2.2 ([10],[12]) *Let $\psi(x,t)$ be the Musielak-Orlicz function complementary to $\varphi(x,t)$ in the Young sense with respect to the variable s, such that:*

$$\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\}.$$

Remark 2.3 ([10]) *The Musielak-Orlicz function $\varphi(x,t)$ satisfies the Δ_2-condition if there exist $k > 0$ and a nonnegative function $h(\cdot) \in L^1(\Omega)$ such that:*

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \quad a.e. x \in \Omega,$$

for large values of t, or for all values of t.

2.2 Musielak-Orlicz Spaces

The measurability of a function $u : \Omega \mapsto R$ refers to Lebesgue measurability. Consider the functional:

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx,$$

where $u : \Omega \mapsto R$ is a measurable function, and define the set:

$$K_\varphi(\Omega) = \{u : \Omega \longmapsto R measurable \mid \varrho_{\varphi,\Omega}(u) < +\infty\},$$

called the Musielak-Orlicz class. The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, equivalently:

$$L_\varphi(\Omega) = \left\{u : \Omega \longmapsto R measurable \mid \varrho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < +\infty, for some \lambda > 0\right\}.$$

In $L_\varphi(\Omega)$, we consider two equivalent norms. The Luxemburg norm:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0 \mid \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right)dx \leq 1\right\},$$

and the Orlicz norm:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\psi,\Omega}\leq 1}\int_\Omega |u(x)v(x)|dx,$$

where $\psi(x,t)$ is the complementary Musielak-Orlicz function to $\varphi(x,t)$. The generalized Hölder inequality ([2]) is given by:

$$\left|\int_\Omega uvdx\right| \leq 2\|u\|_{\varphi,\Omega}\|v\|_{\psi,\Omega} \quad for any \quad u \in L_\varphi(\Omega) and v \in L_\psi(\Omega).2.1$$

The closure in $L_\varphi(\Omega)$ of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. This is a separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if $\varphi(x,t)$ satisfies the Δ_2-condition for large values of t, or for all values of t.

2.3 Musielak-Orlicz-Sobolev Spaces

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $D^\alpha u$ denote distributional derivatives. The Musielak-Orlicz-Sobolev space $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) consists of all measurable functions u such that u and its distributional derivatives up to order 1 lie in $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$).

Define the convex modular on $W^1L_\varphi(\Omega)$ by:

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1} \varrho_{\varphi,\Omega}(D^\alpha u).$$

The norm on $W^1L_\varphi(\Omega)$ is given by:

$$\|u\|_{1,\varphi,\Omega} = \inf\left\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\} \quad foranyu \in W^1L_\varphi(\Omega).$$

The pair $(W^1L_\varphi(\Omega), \|u\|_{1,\varphi,\Omega})$ is a Banach space if the Musielak function $\varphi(x,\cdot)$ satisfies:

$$thereexistsaconstantc > 0suchthat \inf_{x\in\Omega} \varphi(x,1) \geq c.2.2$$

The spaces $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) can be identified with subspaces of the product of $N+1$ copies of $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$), denoted by $\Pi L_\varphi(\Omega)$ (resp. $\Pi E_\varphi(\Omega)$). We will use the weak topologies $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$ and $\sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega))$.

The space $W_0^1E_\varphi(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_\varphi(\Omega)$, and the space $W_0^1L_\varphi(\Omega)$ as the weak $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$ closure of $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$. For further details on Musielak-Orlicz-Sobolev spaces, we refer to [2].

Define the following spaces of distributions:

$$W^{-1}L_\psi(\Omega) = \{f \in \mathcal{D}'(\Omega) : \ldots\}.$$

3 Technical Lemmas

This section presents lemmas essential for proving the main existence theorem.

Lemma 3.1 ([10]) *Let Ω be an open bounded subset of R^N satisfying the segment property. If $u \in (W_0^1L_\varphi(\Omega))^N$, then:*

$$\int_\Omega div(u)dx = 0.$$

Lemma 3.2 ([5]) *Let Ω be a bounded Lipschitz domain in R^N and let φ and ψ be two complementary Musielak-Orlicz functions satisfying: (a) There exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geq c$, (b) There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$:*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log(|x-y|)}\right)} \quad forallt \geq 1, 3.1$$

(c) $\int_\Omega \varphi(x,1)dx < \infty$, 3.2 (d) There exists a constant $C > 0$ such that $\psi(x,1) < C$ a.e. in Ω. 3.3

Under these assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for modular convergence, and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ for modular convergence.

Lemma 3.3 ([15],[13]) *Let Ω be a bounded Lipschitz domain of R^N and let φ be a Musielak-Orlicz function satisfying:*

$$\int_0^\infty \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}}dt = \infty \quad and \quad \int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}}dt < \infty.3.4$$

Define a function $\varphi_*^{-1} : \Omega[0,\infty) \to [0,\infty)$ *by:*

$$\varphi_*^{-1}(x,s) = \int_0^s \frac{\varphi_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} d\tau \quad for x \in \Omega and s \in [0,\infty).$$

Under the conditions of Lemma 3.2:

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_{\varphi_*}(\Omega),$$

where φ_* *is the Sobolev conjugate function of* φ. *Moreover, if* σ *is any Musielak function increasing essentially more slowly than* φ_* *near infinity, then the embedding:*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_\sigma(\Omega),$$

is compact.

Lemma 3.4 ([10]) *(Poincaré inequality) Let* Ω *be a bounded Lipschitz domain of* R^N *and let* φ *be a Musielak-Orlicz function satisfying the conditions of Lemma 3.3. Then there exists a constant* $C > 0$ *such that:*

$$\|u\|_\varphi \le C\|\nabla u\|_\varphi \qquad \forall u \in W_0^1 L_\varphi(\Omega).$$

Lemma 3.5 ([9]) *Let* Ω *be a bounded Lipschitz domain of* R^N *and let* φ *be a Musielak-Orlicz function satisfying condition (3.1). Assume also that* φ *depends only on* $N-1$ *coordinates of* x. *Then there exists a constant* $\lambda > 0$ *depending only on* Ω *such that:*

$$\int_\Omega \varphi(x,|v|)dx \le \int_\Omega \varphi(x,\lambda|\nabla v|)dx \quad for all v \in W_0^1 L_\varphi(\Omega).$$

Lemma 3.6 ([14],[15]) *Let* $u \in L_\varphi(\Omega)$ *and* $u_n \in L_\varphi(\Omega)$ *with* $\|u_n\|_{\varphi,\Omega} \le C$. *If* $u_n(x) \to u(x)$ *a.e. in* Ω, *then* $u_n \rightharpoonup u$ *in* $L_\varphi(\Omega)$ *for* $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$.

Lemma 3.7 ([15]) *Let* $F : R \mapsto R$ *be a uniformly Lipschitz function with* $F(0) = 0$. *Let* $\varphi(x,\cdot)$ *be a Musielak-Orlicz function and* $u \in W_0^1 L_\varphi(\Omega)$. *Then* $F(u) \in W_0^1 L_\varphi(\Omega)$. *Moreover, if the set* D *of discontinuity points of* $F'(\cdot)$ *is finite, then:*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. in \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e. in \{x \in \Omega : u(x) \in D\}. \end{cases} \quad 3.5$$

Lemma 3.8 ([1]) *Let* $u_n, u \in L_\sigma(\Omega)$. *If* $u_n \to u$ *with respect to modular convergence, then* $u_n \to u$ *for* $\sigma(L_\sigma(\Omega), L_\psi(\Omega))$.

Lemma 3.9 ([6],[4]) *Under the assumptions of Lemma 3.2, and assuming that* $\varphi(x,t)$ *depends only on* $N-1$ *coordinates of* x, *there exists a constant* $\overline{C_1} > 0$ *depending only on* Ω *such that:*

$$\int_\Omega \varphi(x,|u|)dx \le \int_\Omega \varphi(x, C_1|\nabla u|)dx.$$

3.7

Definition 3.1 *Let* φ *be a Musielak-Orlicz function. Define the set:*

$$A_\varphi = \left\{\sigma : \Omega R_+ \longrightarrow R_+ \middle| \sigma \prec\prec \varphi and \int_0^1 \varphi\left(x, \beta g^{-1}\left(x, \frac{1}{r^{1-\frac{1}{N}}}\right)\right) dx < \infty a.e. in \Omega \right\},$$

where $H(x,r) = \frac{\varphi(x,r)}{r}$.

Lemma 3.10 *Let Ω be an open subset of R^N with finite measure. Let φ be a Musielak-Orlicz function under assumption (3.6) and the assumptions of Lemma 3.2. For any $u \in W_0^1 L_\varphi(\Omega)$ such that $\int_\Omega \varphi(x, |\nabla u|)dx < \infty$, we have for all $x \in \Omega$:*

$$-\mu'(t) \geq N C_N^{\frac{1}{N}} \mu(t)^{1-\frac{1}{N}} C\left(x, \frac{-1}{C_N^{\frac{1}{N}} \mu(t)^{1-\frac{1}{N}}} \frac{d}{dt} \int_{|u|>t} \varphi(s, |\nabla u|)ds\right), 3.8$$

for a.e. $t > 0$. Here μ is the distribution function of u, and the function $C(\cdot, \cdot)$ is defined by:

$$C(x,t) = \frac{1}{g_x^{-1}(x,t)} \quad with \quad H(x,t) = \frac{\varphi(x,t)}{t},$$

C_N is the measure of the unit ball of R^N, and $\mu(t) = meas\{|u| > t\}$.

Proof ([2]): By definition of the Musielak-Orlicz function, φ is an increasing convex function in t, then K is an increasing convex function in t, and $C(\cdot, \cdot)$ is a decreasing convex function in t.

Lemma 3.11 ([2]) *Let φ be a Musielak-Orlicz function under assumption (3.6) and the assumptions of Lemma 3.2, and $\sigma \in A_\varphi$ with $\sigma \sim \varphi$. There exists a constant $B \leq 1$ such that:*

$$\frac{d}{dt} \int_{|u|<t} \sigma(s, |\nabla u|)ds \leq -\mu'(t)\sigma\left(x, B g_x^{-1}\left(\frac{1}{N C_N^N \mu(t)^{1-\frac{1}{N}}} \frac{d}{dt} \int_{|u|<t} \varphi(s, |\nabla u|)ds\right)\right), 3.9$$

for each $x \in \Omega$ and for any $u \in W_0^1 L_\varphi(\Omega)$ such that $\int_\Omega \varphi(x, |\nabla u|)dx < \infty$.

4 Essential Assumptions

Let Ω be a bounded open subset of R^N $(N \geq 2)$, and $\varphi(x,t)$ be a Musielak-Orlicz function. Let $\psi(x,t)$ be the complementary Musielak-Orlicz function to $\varphi(x,t)$, satisfying the conditions of Lemma 3.2. Let $\gamma(x,t)$ be a Musielak-Orlicz function such that $\gamma \prec\prec \varphi$.

Consider a Leray-Lions operator $A : D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_\psi(\Omega)$ given by:

$$A(u) = -diva(x, u, \nabla u),$$

where $a : \Omega R R^N \to R^N$ is a Carathéodory function (measurable in x for every (s, ξ), and continuous in ξ for almost every x) satisfying:

$$|a(x,s,\xi)| \leq k_1(c(x) + \psi_x^{-1}(\gamma(x, k_2|s|)) + \psi_x^{-1}(\varphi(x, k_3|\xi|))), 4.1$$

$$(a(x,s,\xi) - a(x,s,\xi^*)) \cdot (\xi - \xi^*) > 0 \quad for \xi \neq \xi^*, 4.2$$

$$a(x,s,\xi) \cdot \xi \geq \alpha \cdot \varphi(x, |\xi|), 4.3$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in R R^N$, where $c(x)$ is a nonnegative function in $E_\psi(\Omega)$, and $\alpha, \lambda > 0$, $k_1, k_2, k_3 \geq 0$.

The nonlinear terms $g(x,s,\xi)$ and $\sigma(x,s)$ are Carathéodory functions satisfying:

$$|\sigma(u)| \geq 0, 4.4$$

$$|g(x,s,\xi)| \leq c(x) + \ell(s)\varphi(x, |\xi|), 4.5$$

where $\ell : R^+ \longmapsto R^+$ is a continuous positive function in $L^1(R)$, and $d(x) \in L^1(\Omega)$.

$$\mu \in M_b(\Omega), 4.6$$

$$\sigma \circ g^{-1} is a Musielak-Orlicz function. 4.7$$

5 Main Results

Let Ω be an open bounded subset of R^N $(N \geq 2)$, and let φ and ψ be two complementary Musielak-Orlicz functions. Define the set:

$$\tau_0^{1,\psi}(\Omega) = \{u : \Omega \longrightarrow R measurable \mid T_k(u) \in D(A)\}.$$

Theorem 5.1 *Assume that conditions (4.1)-(4.7) hold, with $A_\varphi \neq \emptyset$. Then there exists at least one solution to the following problem:*

$$\begin{cases} u \in \tau_0^1(\Omega) \cap W_0^1 L_\sigma(\Omega) \quad \forall \sigma \in A_\varphi, \\ \langle A(u), v\rangle + \int_\Omega g(x,u,\nabla u) v dx + \langle \sigma(u), v\rangle = \langle \mu, v\rangle \quad \forall v \in D(A). \end{cases} \quad 5.1$$

Proof of Theorem 5.1
Step 1: Existence of Weak Solutions for Approximate Problems
Consider the following approximate equation for any $n \in N$:

$$\int_\Omega [a(x,u,\nabla u)\nabla v + g_n(x,u,\nabla u)v + \sigma(u_n)v]dx = \int_\Omega \mu_n v dx \quad \forall v \in W_0^1 L_\sigma(\Omega), 5.2$$

where $g_n(x,u,\nabla u) = \frac{g(x,u,\nabla u)}{1+\frac{1}{n}|g(x,u,\nabla u)|}$ and $(\mu_n)_n \in \mathcal{D}(\Omega)$ is a sequence such that:

$$\mu_n \longrightarrow \mu. 5.3$$

in the sense of distributions.

We prove that for every n, there exists at least one solution u_n of (5.2) with $u_n \in W_0^1 E_\sigma(\Omega)$.

Proposition 5.1 ([8],[11]) *Let φ and ψ be two complementary Musielak-Orlicz functions satisfying the conditions of Lemma 3.2. Assume that (4.1)-(4.6) hold. Then, for any $n \in N$, there exists at least one solution $u_n \in W_0^1 E_\sigma(\Omega)$ of (5.2).*

Step 2: A Priori Estimates
Consider the approximate problems:

$$\begin{cases} u_n \in \tau_0^1(\Omega) \cap W_0^1 E_\sigma(\Omega) \quad \forall \sigma \in A_\varphi, \\ \langle A(u_n), v\rangle + \int_\Omega g_n(x,u,\nabla u) v dx + \langle \sigma(u_n), v\rangle = \langle \mu_n, v\rangle \quad \forall v \in W_0^1 L_\sigma(\Omega). \end{cases} \quad 5.4$$

By Proposition 5.1, there exists at least one solution u_n of (5.4).

Lemma 5.1 *Let u_n be a solution of the approximate problem (5.2). Then:*
1. For all $k > 0$, there exists a constant C (independent of n and k) such that:

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) dx \leq C_2 k, 5.5$$

$$\int_\Omega \varphi(x, |\nabla T_k(u_n)|) dx \leq C_3 k, 5.6$$

$$\int_\Omega \sigma(T_k(u_n))\nabla T_k(u_n) dx = \int_\Omega div\psi(u_n) = 0.5.7$$

2. There exists a measurable function u such that:

$$u_n \longrightarrow u \quad a.e. in \Omega. 5.8$$

3.

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi \quad weakly in (L_\psi(\Omega))^N for \sigma(\Pi L_\psi, \Pi E_\varphi). 5.9$$

Proof of Lemma 5.1:
1. Let

$$(v_0 \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$$

ith $v_0 \geq 0$. Taking $v = \exp(G(u_n))v_0$ as a test function in (5.2), where $G(s) = \int_0^s \frac{1}{\alpha}\ell(r)dr$, we btain:

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\frac{\ell(u_n)}{\alpha}\nabla u_n v_0 dx + \int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx + \int_\Omega g_n(x,u_n,\nabla u_n)\exp(G(u_n))v_0 dx + \int_\Omega \sigma(u_n)\exp(G(u_n))v_0 dx = \int_\Omega \mu_n \exp(G(u_n))v_0 dx.$$

.10

Using (4.3) and (4.5), we simplify by the term $\int_\Omega \ell(u_n)\varphi(x,|\nabla u_n|)v_0 dx$, and obtain:

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx + \int_\Omega \sigma(u_n)\exp(G(u_n))v_0 dx \leq \int_\Omega \mu_n \exp(G(u_n))v_0 dx + \int_\Omega c(x)\exp(G(u_n))v_0 dx.5.1$$

By (4.4):

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx \leq \int_\Omega \mu_n \exp(G(u_n))v_0 dx + \int_\Omega c(x)\exp(G(u_n))v_0 dx.5.12$$

Taking $v = \exp(-G(u_n))v_0$ as a test function in (5.2), we deduce:

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(-G(u_n))\nabla v_0 dx + \int_\Omega c(x)\exp(-G(u_n))v_0 dx \geq \int_\Omega \mu_n \exp(-G(u_n))v_0 dx.5.13$$

Choosing $v_0 = T_k(u_n)^+$ in (5.12):

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+ dx \leq \int_\Omega \mu_n \exp(G(u_n))T_k(u_n)^+ dx + \int_\Omega c(x)\exp(G(u_n))T_k(u_n)^+ dx.$$

Since $\ell \in L^1(R)$, we have $G(-\infty) \leq G(s) \leq G(+\infty)$ and $|G(\pm\infty)| \leq \frac{1}{\alpha}\|\ell\|_{L^1(R)}$. Then:

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)^+)\nabla T_k(u_n)^+ dx \leq \exp\left(\frac{\|\ell\|_{L^1(R)}}{\alpha}\right) k\left[\|\mu\|_{M_b(\Omega)} + \|b\|_{L^1(\Omega)}\right] = kC_4.$$

Using (4.3):

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)^+|)dx \leq kC_5.$$

Choosing $v_0 = T_k(u_n)^-$ in (5.13), we similarly obtain:

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)^-|)dx \leq kC_6.$$

Results (5.5), (5.6), and (5.7) follow accordingly.
2. Using (3.7):

$$\inf_{x\in\Omega}\varphi\left(x,\frac{k}{c_1}\right) meas\{|u_n|>k\} \leq \int_{|u_n|>k}\varphi\left(x,\frac{|T_k(u_n)|}{c_1}\right)dx \leq \int_\Omega \varphi(x,|\nabla T_k(u_n)|)dx \leq kC_7.$$

Thus:

$$meas\{|u_n|>k\} \leq \frac{kC_7}{\inf_{x\in\Omega}\varphi(x,\frac{k}{c_1})} \quad forallnandk.$$

Assume there exists a positive function M such that $\lim_{t\to\infty}\frac{M(t)}{t} = +\infty$ and $M(t) = ess\,inf_{x\in\Omega}\varphi(x,t)$ or all $t \geq 0$. Then:

$$\lim_{t\to\infty} meas\{|u_n|>k\} = 0.$$

By property (1) of Lemma 5.1, $T_k(u_n)$ is bounded in $W_0^1L_\varphi(\Omega)$. Thus, there exists $T_k \in W_0^1L_\varphi(\Omega)$ such that:

$$T_k(u_n) \longrightarrow T_k \quad weaklyinW_0^1L_\varphi(\Omega)for\sigma(\Pi L_\varphi, \Pi E_\psi), stronglyinE_\varphi(\Omega), anda.e.in\Omega.$$

By (2) of Lemma 5.1, the sequence $(u_n)_n$ converges almost everywhere to some measurable function u. Therefore:

$$T_k(u_n) \longrightarrow T_k(u) \quad weaklyinW_0^1L_\psi(\Omega)for\sigma(\Pi L_\varphi, \Pi E_\psi), stronglyinE_\varphi, anda.e.in\Omega.$$

3. To prove that $a(x, T_k(u_n), \nabla T_k(u_n))_n$ is bounded in $(L_\psi(\Omega))^N$ for all $k > 0$, let $\in (E_\varphi(\Omega))^N$ be arbitrary. By (4.1):

$$(a(x, u_n, \nabla u_n) - a(x, u_n,))(\nabla u_n -) \geq 0.$$

Then:

$$\int_{|u_n|\leq k} a(x, u_n, \nabla u_n)dx \leq \int_{|u_n|\leq k} a(x, u_n, \nabla u_n)\nabla u_n dx + \int_{|u_n|\leq k} a(x, u_n,)(-\nabla u_n)dx.$$

By (4.1) and Remark 2.1, there exists $k' > 0$ such that $\gamma(x, v_1 k) \leq k'\varphi(x, 1)$, and for $\lambda > 0$ large enough:

$$\int_{|u_n|\leq k} \psi\left(\frac{|a(x, u_n, \frac{}{v_2})|}{3\beta}\right) dx \leq \frac{1}{3}\left[\int_\Omega \psi(c(x))dx + \int_\Omega k \cdot \varphi(x, 1)dx + \int_\Omega \varphi(x, ||)dx\right] \leq C_7.5.14$$

Thus, $a(x, T_k(u_n), \frac{}{v_2})$ is bounded in $(L_\psi(\Omega))^N$. By (5.14), (5.5), and the Banach-Steinhaus theorem, the sequence $a(x, T_k(u_n), \nabla T_k(u_n))$ remains bounded in $(L_\psi(\Omega))^N$. For a subsequence:

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x, T_k(u), \nabla T_k(u)) \quad in(L_\psi(\Omega))^N for\sigma(\Pi L_\varphi, \Pi E_\psi).$$

Step 3: Almost Everywhere Convergence of the Gradients
To establish almost everywhere convergence of the gradients, we prove the following proposition.

Proposition 5.2 *Let $(u_n)_n$ be a solution of the approximate problem (5.2). Then:*
1.

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{-(m+1)\leq u_n\leq -m\}} a(x, u_n, \nabla u_n)\nabla u_n dx = 0.5.15$$

2. For a subsequence, as $n \to \infty$:

$$\nabla u_n \to \nabla u \quad a.e.in\Omega.5.16$$

Proof:
1. Take $v_0 = T_1(u_n - T_m(u_n))^-$ in (5.5). This function is admissible since $v_0 \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$ and $v_0 \geq 0$. Then:

$$-\int_\Omega a(x, u_n, \nabla u_n)\exp(-G(u_n))\nabla T_1(u_n - T_m(u_n))^- dx \leq \int_\Omega c(x)\exp(-G(u_n))T_1(u_n - T_m(u_n))^- dx.$$

Since μ is nonnegative:

$$\int_{\{-(m+1)\leq u_n\leq -m\}} a(x, u_n, \nabla u_n)\exp(-G(u_n))\nabla u_n dx \leq \exp\left(\frac{\|\ell\|_{L^1(R)}}{\alpha}\right)\int_\Omega |c(x)|T_1(u_n - T_m(u_n))^- dx.$$

By Lebesgue's theorem, we conclude result (5.15).
2. To show that $\nabla u_n \longrightarrow \nabla u$ a.e. in Ω, adapt the proof from [16] following the same steps with $\Psi = 0$.
Step 4: Equi-integrability of the Nonlinearity Sequence

We prove that:

$$g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \quad stronglyinL^1(\Omega).5.17$$

Consider $v_0 = \int_0^{u_n} p(s)\chi_{\{s>h\}}ds$ in (5.12). We get:

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx \le \int_\Omega \mu_n \exp(G(u_n))v_0 dx + \int_\Omega c(x)\exp(G(u_n))v_0 dx.$$

Using (4.3) and (4.5):

$$\alpha \int_{\{u_n>h\}} \ell(u_n)\varphi(x,\nabla u_n)dx \le \left(\int_h^{+\infty} \ell(s)ds\right)\exp\left(\frac{\|\ell\|_{L^1(R)}}{\alpha}\right)\left[\|\mu\|_{M(\Omega)} + \|b\|_{L^1(\Omega)}\right],$$

$$\int_{\{u_n>h\}} \ell(u_n)\varphi(x,\nabla u_n)dx \le \frac{C_4}{\alpha}\int_h^{+\infty} \ell(s)ds.$$

Since $\ell \in L^1(R)$:

$$\lim_{h\to\infty}\sup_{n\in N}\int_{\{u_n>h\}} \ell(u_n)\varphi(x,\nabla u_n)dx = 0.5.18$$

Similarly, taking $v_0 = \int_0^{u_n} p(s)\chi_{\{s<-h\}}ds$ in (5.13):

$$\lim_{h\to\infty}\sup_{n\in N}\int_{\{u_n<-h\}} \ell(u_n)\varphi(x,\nabla u_n)dx = 0.$$

We conclude:

$$\lim_{h\to\infty}\sup_{n\in N}\int_{\{|u_n|>h\}} \ell(u_n)\varphi(x,\nabla u_n)dx = 0.$$

Let $D\subset\Omega$. Then:

$$\int_D \ell(u_n)\varphi(x,\nabla u_n)dx \le \max_{|s|\le h}\ell(s)\int_{D\cap\{|u_n|\le h\}}\varphi(x,\nabla u_n)dx + \int_{D\cap\{|u_n|>h\}}\ell(u_n)\varphi(x,\nabla u_n)dx.$$

Consequently, $\ell(u_n)\varphi(x,\nabla u_n)$ is equi-integrable. Since $\ell(u_n)\varphi(x,\nabla u_n)$ converges to $\ell(u)\varphi(x,\nabla u)$ strongly in $L^1(\Omega)$, we obtain the result.

Step 5: Verification that u Satisfies (5.1)

- u_n is bounded in $W_0^1L_\sigma(\Omega)$ and converges to u strongly in $L_\sigma(\Omega)$, where $\sigma \in A_\varphi$.

First, take $T_\epsilon(u_n - T_t(u_n))$, $\epsilon>0$, $t>0$ as a test function in (5.4). From (4.5) and (5.18):

$$\int_{\{t\le|u_n|\le t+1\}} a(x,u_n,\nabla u_n)dx \le \epsilon C_{10}.$$

The constant C_{10} is independent of n, ϵ, and t. Then:

$$\frac{1}{\epsilon}\int_{\{t\le|u_n|\le t+1\}}\varphi(x,\nabla u_n)dx \le \frac{C_{10}}{\alpha\epsilon}.$$

Letting $\epsilon\longrightarrow 0$:

$$-\frac{d}{dt}\int_{\{|u_n|>t\}}\varphi(x,\nabla u_n)dx \le \frac{C_{10}}{\alpha}.5.19$$

Second, let $\sigma\in A_\varphi$ with $\sigma\sim\varphi$. Using Lemmas 3.10 and 3.11, equation (5.18), and techniques from [10], we deduce that ∇u_n is bounded in $L_\sigma(\Omega)$ for each $\sigma\in A_\varphi$. Thus, u_n is bounded in $W_0^1L_\sigma(\Omega)$ for each $\sigma\in A_\varphi$.

- $a(x,u_n,\nabla u_n)\longrightarrow a(x,u,\nabla u)$ weakly for $\sigma(\Pi L_{\sigma\circ g^{-1}},\Pi E)$, where and $\sigma\circ g^{-1}$ are two complementary Musielak-Orlicz functions.

Using (4.1) and Remark 2.1:

$$\int_\Omega \sigma\circ g^{-1}\left(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\right)dx \leq \int_\Omega \sigma\circ g^{-1}\left(\frac{1}{6}\left[c(x)+k(v_1)\psi_x^{-1}(\varphi(x,|u_n|))+\psi_x^{-1}(\varphi(x,v_2|\nabla u_n|))\right]\right)dx$$

Since $\sigma\circ g^{-1}$ is a Musielak-Orlicz function:

$$\int_\Omega \sigma\circ g^{-1}\left(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\right)dx \leq \frac{1}{3}\int_\Omega \left[\sigma\circ g^{-1}\left(\frac{1}{2}c(x)\right)+\sigma\circ g^{-1}\left(\frac{1}{2}k(v_1)\psi_x^{-1}(\varphi(x,|u_n|))\right)+ \sigma\circ g^{-1}\left(\psi_x^{-1}\left(\frac{1}{2}\varphi(x,v_2|\nabla u_n|)\right)\right)dx.\right.$$

.20

By definition of the Musielak-Orlicz function:

$$\frac{1}{2}\psi_x^{-1}(\varphi(x,t)) \leq \frac{\varphi(x,t)}{t}.$$

By definition of H:

$$\varphi(x,g_x^{-1}(t/2)) \leq \psi(x,t).$$

Hence, by Remark 2.1:

$$\sigma\circ g^{-1}\left(\frac{1}{2}c(x)\right) \leq k_1 g^{-1}\left(\frac{1}{2}c(x)\right) \leq k_1\psi(x,c(x)).5.21$$

Also:

$$\sigma\circ g^{-1}\left(\frac{1}{2}\psi_x^{-1}(k(v_1)\varphi(x,|u_n|))\right) \leq \sigma\circ g^{-1}\left(\frac{1}{2}\psi_x^{-1}(k(v_1)\varphi(x,k_2|u_n|))\right) \leq \sigma\circ g^{-1}\left(\frac{\varphi(x,k_2|u_n|)}{v_2|\nabla u_n|}\right),5.22$$

where $k_2=\max(1,k(v_1))$. Then:

$$\sigma\circ g^{-1}\left(\frac{1}{2}\psi_x^{-1}(k(v_1)\varphi(x,|u_n|))\right) \leq k_3\varphi(x,|u_n|).5.23$$

And:

$$\sigma\circ g^{-1}\left(\frac{1}{2}\psi_x^{-1}(k(v_1)\varphi(x,v_2|\nabla u_n|))\right) \leq \sigma\circ g^{-1}\left(\frac{\varphi(x,v_2|\nabla u_n|)}{v_2|\nabla u_n|}\right) = \sigma(x,v_2|\nabla u_n|).5.24$$

Using Remark 2.1:

$$\sigma\circ g^{-1}\left(\frac{1}{2}\psi_x^{-1}(k(v_1)\varphi(x,v_2|\nabla u_n|))\right) \leq k_4\varphi(x,|\nabla u_n|).5.25$$

Applying (5.21), (5.23), and (5.25) in (5.20):

$$\frac{1}{3}\int_\Omega \sigma\circ g^{-1}\left(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\right)dx \leq \int_\Omega [k_1\psi(x,c(x))+k_3\varphi(x,|u_n|)+k_4\varphi(x,|\nabla u_n|)]\,dx < C_{11}.$$

Consequently, $a(x,u_n,\nabla u_n) \longrightarrow a(x,u,\nabla u)$ weakly for $\sigma(\Pi L_{\sigma\circ g^{-1}},\Pi E)$.
- Taking $v\in\mathcal{D}(\Omega)$ as a test function in the approximate equation (5.2):

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla v dx + \int_\Omega g(x,u_n,\nabla u_n)v dx + \int_\Omega div\sigma(u)v dx = \int_\Omega \mu_n v dx.$$

Since $u_n \longrightarrow u$ strongly in $(E_k(\Omega))^N$ for every $H \prec\prec \sigma$, for all $\sigma\in A_\varphi$, using (5.16) and (5.3), we can pass to the limit as $n \longrightarrow +\infty$ to complete the proof of Theorem 5.1.

References

1. A. Abergi, J. Bennouna, and M. Elmassoudi. Nonlinear Elliptic Equation with measure data in Musielak-Orlicz-Sobolev spaces. Gulf J. Math. 6(4) (2018), 79-100.
2. A. Abergi, J. Bennouna, and M. Elmassoudi. Nonlinear Elliptic Equation with some measure data in Musielak-Orlicz-Sobolev spaces. Math. Comput. Sci. 19(2) (2019), 227-242.
3. M. L. Ahmed Oubeid, A. Benkirane, M. Sidi El Vally. Nonlinear Elliptic Equations Involving Measure Data in Musielak-Orlicz-Sobolev Spaces. J. Abstr. Differ. Equ. Appl. 4(1) (2013), 43-57.
4. T. Ahmedatt, A. Abergi, A. Touzani, and C. Yazough. On some nonlinear hyperbolic $p(x,t)$-Laplacian equations. J. Appl. Anal. 24(1) (2018), 79-100.
5. M. Ait Khellou, A. Benkirane, S. M. Douiri. Existence of solutions for elliptic equations having natural growth terms in Musielak spaces. J. Math. Comput. Sci. 4(4) (2014), 665-688.
6. M. Ait Khellou, A. Benkirane. Strongly non-linear elliptic problems in Musielak spaces with L^1 data. Nonlinear Studies 23(3) (2016), 491-510.
7. M. Ait Khellou, A. Benkirane, and S. M. Douiri. Inequality of type Poincaré in Musielak spaces and applications to some nonlinear elliptic problems with L^1 data. Complex Variables and Elliptic Equations. 60(2015), 1217-1242.
8. Y. Akdim, N. El gorch, and M. Mekkour. Existence of renormalized solutions for $p(x)$-parabolic equations with three unbounded nonlinearities. Bol. Soc. Paran. Mat. 34(1) (2016), 225-252.
9. M. Al-Hawmi, A. Benkirane, H. Hijaj, and A. Touzani. Existence and Uniqueness of Entropy Solutions for some Nonlinear Elliptic Unilateral Problems in Musielak-Orlicz-Sobolev Spaces. Ann. Univ. Craiova Math. Comput. Sci. Ser. 44(1) (2017), 1-20.
10. M. Al-Hawmi and M. Al-Hasisi. Existence of renormalized solutions for nonlinear elliptic problem in Musielak-Orlicz-Sobolev spaces. J. Math. Comput. Sci. 19(2) (2022), 227-242.
11. S. Antontsev and S. Shmarev. Evolution PDEs with Nonstandard Growth Conditions: Existence, Uniqueness, Localization, Blow-up. Atlantis Studies in Differential Equations, vol. 4. Series editor Michel Chipot. Zurich, Switzerland, 2015.
12. A. Benkirane, A. Elmahi. Almost everywhere convergence of the gradients of solutions to elliptic equations in Orlicz spaces and application. Nonlinear Anal. Theory Methods Appl. 28 (1997), 1769-1784.
13. A. Benkirane, M. Sidi El Vally (Ould Mohamedhen Val). Some approximation properties in Musielak-Orlicz-Sobolev spaces. Thai J. Math. 10(2) (2012), 371-381.
14. A. Benkirane, M. Sidi El Vally (Ould Mohamedhen Val). An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces. Bull. Belg. Math. Soc. Simon Stevin 20(1) (2013), 57-75.
15. A. Benkirane, M. Sidi El Vally. Variational inequalities in Musielak-Orlicz-Sobolev spaces. Bull. Belg. Math. Soc. Simon Stevin 21(5) (2014), 787-811.
16. G. Dong and X. Fang. Existence results for some nonlinear elliptic equations with measure data in Orlicz-Sobolev spaces. Boundary Value Problems (2015), 18.