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إثبات وجود الحلول الكونية في الأوساط غير الخطية القوية في فضاءات أورلكس الموزونة

Proving the Existence of Entropy Solutions in Strongly Nonlinear Settings in Weighted Orlicz Spaces

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الملخص

تتناول هذه الورقة البحثية دراسة وجود الحلول الكونية لمسائل إهليلجية غير الخطية بقوة تتميز بشروط نمو غير قياسية. تُصاغ المسألة المدروسة بواسطة مؤثرات من نوع التباعد تتضمن حدودًا غير خطية موزونة ومؤثرات من الرتبة الدنيا، مع معطيات تنتمي فقط إلى الفضاء $L^1(\Omega)$.

يُجرى التحليل ضمن إطار فضاءات أورلكس الموزونة $W_0^1 L_M(\Omega,\rho)$ والتي توفر بيئة طبيعية لتمثيل سلوكيات النمو غير المتجانسة والمعتمدة على الموقع والاتجاه. وبافتراض أن دالة أورلكس تحقق شرط (Δ_2) ، يتم إثبات وجود حلول كونية دون فرض قيود النمو متعددة الحدود التقليدية على الحد اللاخطي بالنسبة لمتغير التدرج.

تعتمد البرهنة على تقنيات قطع مناسبة، ومبرهنات التراص ، ونتائج إدماج دقيقة في فضاءات أورلكس الموزونة. وتُمكّن هذه الأدوات من تجاوز نقص انتظام حد المصدر وقوة لاخطية المؤثر.

تُعمّم النتائج المتحصَّل عليها وتوحّد عدة م برهنات معروفة حول الوجود لمسائل إهليلجية في فضاءات أورلكس وسوبوليف الكلاسيكية، كما تسهم في تعزيز الأساس النظري للمعادلات الإهليلجية اللاخطية الناشئة في النماذج التطبيقية ذات الخصائص المادية غير المتجانسة وغير المنتظمة.

ويمثل هذا العمل تعميمًا لعمل سابق، وذلك في غياب الحد $g(x, s,\varepsilon)$ حيث يُفترض فقط أنه يحقق شروط نمو غير قياسية بالنسبة إلى $|\varepsilon|$.

الكلمات المفتاحية: مسألة إهليلجية غير خطية، الحلول الكونية، فضاءات أورلكس الموزونة، الدمج المتراص.

Abstract

This paper investigates the existence of entropy solutions for strongly nonlinear elliptic problems characterized by nonstandard growth conditions. The considered problem is driven by divergence-form operators involving weighted nonlinear terms and lower-order perturbations, with data belonging merely to $(L^1(\Omega))$. . The analysis is carried out within the framework of Orlicz space $(W_0^1 L_M(\Omega,\rho))$, which provide a natural setting for capturing spatially dependent and anisotropic growth behaviors. Under the assumption that the conjugate Orlicz function M(t) satisfies the (Δ_2)- condition. The existence of entropy solutions is established without imposing the standard polynomial growth restrictions on the nonlinear term with respect to the gradient variable. The proof relies on appropriate truncation techniques, compactness arguments, and refined embedding results in weighted Orlicz spaces. These tools allow us to overcome the lack of regularity of the source term and the strong nonlinearity of the operator. The results extend and unify several known existence theorems for elliptic problems in classical Orlicz and Sobolev spaces. And they contribute to the theoretical foundation of nonlinear elliptic equations arising in applied models with heterogeneous and nonuniform material properties. This work is a generalization of the work developed for previous work without the presence of the $g(x, s, \xi)$ satisfies only some nonstandard growth with respect to $|\xi|$.

Keywords: Nonlinear elliptic problem, entropy solutions, weighted Orlicz spaces, compact imbedding.

1 Introduction

In this study deals with existence of entropy solutions to the following nonlinear Dirichlet problem:

$$\begin{cases} A(u) + g(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $A(u) = -div(\rho(x)a(x,u,\nabla u)), \Omega$ is a bounded domain of $I\!R(N \geq 2).a(x,u,\nabla u) = (a_i(x,u,\nabla u))_{1\leq i\leq N}$, $a_i : \Omega \times I\!R \times I\!R^N \longrightarrow I\!R$ is a Carathéodory functions (that is measurable with respect to x in Ω for every $(s,\xi) \in I\!R \times I\!R^N$, and continuous with respect to$(s,\xi) \in I\!R \times I\!R^N$ for almost every$x \in \Omega$) such that for all $\xi, \acute{\xi}$ in $I\!R^N$

$$|(a_i(x,s,\xi)| \leq \phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|s|)) + K_i\overline{M}^{-1}M(c_1|\xi|), \tag{1.2}$$

$$(a(x,s,\xi) - a(x,s,\acute{\xi}))(\xi - \acute{\xi}) \geq 0, \tag{1.3}$$

$$a(x,s,\xi)\xi \geq M(\lambda_1|\xi|), \tag{1.4}$$

where c_1, c_2, λ_1 and K_i belongs to $I\!R_+.M, P$ be two N-functions such that $P \prec\prec M$. Furthermore $\overline{M}, \overline{P}$ be the complementary functions of M and P respectively, ρ be a weight function on Ω (that is, measurable and positive a.e. on Ω) and $\phi_i \in E_{\overline{M}}(\Omega,\rho)$.

In addition, the function $g(x,s,\xi)$ is a Carathéodory function satisfying:

$$g(x,s,\xi)s \geq 0, \tag{1.5}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + M(|\xi|)), \tag{1.6}$$

$$f \in L^1(\Omega). \tag{1.7}$$

The notion of entropy solutions, used in [11], allows us to give a meaning to a possible solution of (1.1). In fact, in [11], Boccardo proved, for $\rho(x) = 1$ and p such that $2 - 1/N < p < N$, the existence and the regularity of an entropy solution u of problem (1.1). For the case that $\rho(x)$=1, $\phi = 0$ and f is a bounded measure, Bénilan et al. [17] proved the existence and the uniqueness of entropy solutions. The same problem is treated by using the notion of entropy solutions introduced in [24], where $\rho(x)$ = 1, $f \in L^1(\Omega)$, and $f \in L^{p'}(\Omega)^N$.
The reason for this, is that $a(x,s,\nabla u)$ does not need to satisfy the strict monotonicity condition that is,

$$\left(a(x,s,\xi) - a(x,s,\acute{\xi})\right)(\xi - \acute{\xi}) > 0, \text{ for all } \xi, \acute{\xi} \in I\!R^N, (\xi \neq \acute{\xi}),$$

of a typical Leray-Lions operator but only a large monotonicity that is

$$\left(a(x,s,\xi) - a(x,s,\acute{\xi})\right)(\xi - \acute{\xi}) \geq 0, \text{ for all } \xi, \acute{\xi} \in I\!R^N,$$

and $g(x,s,\xi)$ presents the nonlinearity of the problem (1.1).
Bensoussan, Boccardo and Murat [16] proved the existence of solutions for the Dirichlet problem of the form (1.1),where $g(x,s,\xi)$ satisfies :

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + |\xi|^p \quad \text{(natural growth condition)},$$

$$g(x,s,\xi).s \geq 0 \qquad \text{(sign condition)}.$$

Further reference is made to [3, 4] for more details. Benkirane and Elmahi [15] have proved the existence theorem of the problem (1.1) in Orlicz-Sobolev space $W^1L_M(\Omega)$, by assuming a sign condition and a natural growth condition on $g(x,s,\xi)$ of the form:

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + M\left(\frac{|\xi|}{\lambda}\right)) \quad (M(.) \text{ is an N-function }).$$

The N-function M(.)is supposed to satisfy the Δ_2-condition and the domain Ω of $I\!R^n$ satisfying the segment property, in order to construct a complementary system

$$(W_0^1 L_M(\Omega), W_0^1 E_M(\Omega); W^{-1} L_{\overline{M}}(\Omega), W^{-1} E_{\overline{M}}(\Omega)).$$

The primary objective of this research is to establish the existence of entropy solutions to equation (1.1) under a weaker assumption known as the large monotonicity condition. This result is derived by utilizing the almost everywhere convergence of the gradients of the approximate equations, a strategy which is particularly relevant given the challenges inherent in directly proving this property within our current framework. Specifically, the standard approach to proving such existence results does not apply in the context of this study. To achieve the desired conclusion, a modified version of Mintys lemma is used, relying on a technique first introduced by Minty [26]. This lemma provides an essential tool for overcoming the difficulties posed by the weaker monotonicity assumption, and its application in this context marks a significant contribution to the broader field of partial differential equations and variational analysis. The structure of this academic article is organized as follows: In Section 2, the key definitions and essential properties associated with the framework of weighted Orlicz-Sobolev spaces are presented. These spaces play a crucial role in the formulation and analysis of the problem at hand. In this section, an abstract theorem forming the basis for the subsequent analysis is presented, and several auxiliary results necessary for proving the main theorem are established. Section 3 is dedicated to stating the main result and providing the detailed proof, which leverages the tools developed in the previous section. Through this approach, a comprehensive and rigorous solution to the problem posed by equation (1.1) is sought.

2 Preliminary

This section provides a comprehensive overview of key definitions and well-established concepts related to N-functions and weighted Orlicz-Sobolev spaces. These topics are extensively examined in the literature, with foundational references available in works such as [1] and [19]. The aim is to outline the core principles and theoretical underpinnings that inform the current understanding and application of these mathematical structures.

2.1 N-function.

Let $M : I\!R^+ \to I\!R^+$ be an N-function, i.e. M is continuous, convex, with

$$M(t) > 0 \quad \text{for} \quad t > 0, \frac{M(t)}{t} \to 0 \quad \text{as} \quad t \to 0 \quad \text{and} \quad \frac{M(t)}{t} \to \infty \quad \text{as} \quad t \to \infty.$$

Equivalently M admits the representation :

$$M(t) = \int_0^t m(\tau)d\tau,$$

where $m : I\!R^+ \to I\!R^+$ is non-decreasing right continuous, with$m(0) = 0, m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function M conjugate to M defined by

$$\overline{M}(t) = \int_0^t \overline{m}(\tau)d\tau,$$

where $\overline{m} : I\!R^+ \to I\!R^+$ is given by $\overline{m}(t) = sup\{s : m \leq t\}$. It is well established that m and $\overline{m}$ may be assumed to be continuous and strictly increasing. The corresponding N-functions are extended to even functions defined over the entire domain $I\!R^+$.

Clearly $\overline{\overline{M}} = M$ and has Young's inequality

$$st \leq M(t) + \overline{M}(s) \quad \text{for all} \quad s, t \geq 0.$$

The N-function M is said to satisfy the Δ_2-condition every-where (resp. infinity) if there exist $k > 0$ (resp. $t_0 > 0$).
Let P and Q be two N-functions, the notation $P \prec\prec Q$ means that P grows essentially less rapidly than Q, that is to say for all $\epsilon > 0$, $\frac{p(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. That is the case if and only if $\frac{Q^{-1}(t)}{P^{-1}(t)} \to 0$ as $t \to \infty$.

2.2 Orlicz-Sobolev space.

Let Ω be an open subset of $I\!R^N$ and M be an N-function. The Orlicz classe $K_M(\Omega)$ (resp. the Orlicz spaces $K_M(\Omega)$)) is the set of all (equivalence classes modulo equality a.e. in Ω of) real-valued measurable functions u defined in Ω and satisfying

$$\int_\Omega M(u(x))dx < \infty \left(resp. \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) dx < \infty \text{ for some } \lambda > 0 \right).$$

$L_M(\Omega)$ is a Banach space equipped the norm:

$$\|u\|_{M,\Omega} = inf \left\{ \lambda > 0 : \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) dx \le 1 \right\}. \tag{2.1}$$

The closure in $L_M(\Omega)$of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$ The usual form is obtained $E_M(\Omega) \subset K_M(\Omega) \subset L_M(\Omega)$)).

The equality $L_M(\Omega = E_M(\Omega)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or note.
The dual of $E_M(\Omega)$ can be identified with $L_M(\Omega)$ by means of the pairing$\int_\Omega u(x)\nu(x)dx$ where $u \in L_M(\Omega)$ and $\nu \in L_{\overline{M}}(\Omega)$ and the dual norm on $L_{\overline{M}}$ is equivalent to $\|.\|_{\overline{M},\Omega}$.
The space $L_M(\Omega)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large, according to whether Ω be infinite measure or note.
Attention is now directed back to the Orlicz-Sobolev spaces $W^1L_M(\Omega)$(resp. $W^1E_M(\Omega)$) is the space of all function u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. in $E_M(\Omega)$). It's Banach space equipped the norm:

$$\|u\|_{1,M} = \sum |D^\alpha u\|_{M,\Omega}. \tag{2.2}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of $\prod L_M$ the weak topology is considered $\sigma(\Pi L_M, \Pi E_M)$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
The space $W_0^1E_M(\Omega)$(resp.$W_0^1L_M(\Omega)$)is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega)$ (resp. $W^1L_M(\Omega)$) for the norm (2.2)(resp. for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

Definition 2.1 *The sequence u_n converges to u in $L_M(\Omega)$ for the modular convergence (denoted by $u_n \to u$ (mod)$L_M(\Omega)$) if*

$$\int_\Omega M\left(\frac{|u_n - u|}{\lambda}\right) dx \to 0 \text{ as } n \to \infty \text{ for some } \lambda > 0.$$

2.3 Weighted Orlicz-Sobolev space.

Let Ω be a domain in $I\!R^N$, M be an N-function and $\rho(x)$ be a weight function on Ω, i.e. measurable positive a.e. on Ω such that:

$$\rho \in L^1(\Omega). \tag{2.3}$$

The weighted Orlicz classe $K_M(\Omega,\rho)$ (resp. the weighted Orlicz space $L_M(\Omega,\rho)$ is the set of all (equivalence classes modulo equality a.e. in Ω) of real-valued measurable functions u defined in Ω and satisfying:
$m_P(u,M)=\int_\Omega M(|u(x)|)\rho(x)dx<\infty$

$$\left(resp.m_p(\frac{u}{\lambda},M)=\int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)\rho(x)dx<\infty \text{ for some } \lambda>0\right).$$

$L_M(\Omega,\rho)$is a Banach space under the norm :

$$\|u\|_{M,\rho}=inf\left\{\lambda>0;m_\rho(\frac{u}{\lambda},M)\leq 1\right\}. \quad (2.4)$$

The closure in $L_M(\Omega,\rho)$of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega,\rho)$ are usually conducted $E_M(\Omega,\rho)\subset K_M(\Omega,\rho)\subset L_M(\Omega,\rho)))$.
The equality $L_M(\Omega,\rho)=E_M(\Omega,\rho)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or note.
Remember that spaces $L_M(\Omega,\rho)$ and $L_{\overline{M}}(\Omega,\rho)$ are naturally associated by duality

$$\langle u,\nu\rangle=\int_\Omega u(x)\nu(x)\rho(x)d(x), u\in L_M(\Omega,\rho), \nu\in L_{\overline{M}}(\Omega,\rho),$$

giving what's called the Orlicz norm on $L_{\overline{M}}(\Omega,\rho)$, which is

$$|||u|||_{M,\rho}=sup\int_\Omega u(x)\nu(x)\rho(x)d(x), m_\rho(\nu,\overline{M}\leq 1),$$

The dual of $E_M(\Omega,\rho)$ can be identified with $L_{\overline{M}}(\Omega,\rho)$ (see [12]). It is easy to prove (as in the case of non-weighted spaces) that

$$\|u\|_{M,\rho}\leq |||u|||_{M,\rho}\leq 2\|u\|_{M,\rho},$$

he space $L_M(\Omega,\rho)$ is reflexive if and only if $M \text{an} \overline{M}$ satisfy the Δ_2-condition for all t or for t large according to whether Ω be infinite measure or note. A return is now made to the weighted Orlicz-Sobolev spaces $W^1L_M((\Omega,\rho))$ (resp. $W^1E_M((\Omega,\rho))$ is the space of all function u such that $u\in L_M(\Omega)$ (resp. $u\in E_M(\Omega)$) and it's distributional derivatives up to order 1 lie in $L_M(\Omega,\rho)$ (resp. $L_M(\Omega,\rho)$) . It's Banach space under the norm

$$\|u\|_{1,M,\rho}=\|u\|_M+\|\nabla u\|_{M,\rho}. \quad (2.5)$$

(Where $\|u\|_M=\|u\|_{M,\Omega}$). Thus $W^1L_M((\Omega,\rho))$ and $W^1E_M((\Omega,\rho))$ can be identified with subspaces of $\Pi L_{M,\rho}=L_M\times\Pi L_M(\Omega,\rho)$ the weak topology is considered $\sigma(\Pi L_{M,\rho},\sigma(\Pi E_{\overline{M}},\rho)$ and $\sigma(\Pi L_{M,\rho},\sigma(\Pi L_{\overline{M}},\rho)$ the space $W_0^1E_M(\Omega,\rho)$ (resp. $W_0^1L_M(\Omega,\rho)$ is defined by the closure of $D(\Omega)$ in $W^1E_M((\Omega,\rho))$ (resp. $W^1L_M((\Omega,\rho)))$ for the norm (2.5)(resp. for the topology $\sigma(\Pi L_{M,\rho},\sigma(\Pi E_{\overline{M}},\rho))$.

Definition 2.2 *The sequence u_n converges to u in $L_M(\Omega,\rho)$ for the modular convergence denoted by $u_n\to u$ (mod) $L_M((\Omega,\rho))$ if $\int_\Omega M\left(\frac{u_n-u}{\lambda}\right)\rho(x)dx\to 0$ as $n\to\infty$ for some $\lambda>0$.*

Definition 2.3 *The sequence u_n converges to u in $W^1L_M((\Omega,\rho)$ for the modular convergence (denoted by $u_n\to u(mod)W^1L_M((\Omega,\rho)$ if for some $\lambda>0$ $\int_\Omega M\left(\frac{u_n-u}{\lambda}\right)dx\to 0$ as $n\to\infty$ and $\int_\Omega M\left(\frac{|D^\alpha(u_n-u)|}{\lambda}\right)\rho(x)dx\to 0$ as $n\to\infty$ for $|\alpha|=1$.*

Lemma 2.1 [10] *Let M be an N-function. If $u_n\in L_M(\Omega)$ converges a.e. to u and u_n bounded in $L_M(\Omega)$, the $u\in L_M(\Omega)$ and $u_n\to u$ for the topology $\sigma(L_M(\Omega),E_{\overline{M}}(\Omega,\rho))$.*

Lemma 2.2 [10] *If the sequence $u_n\in L_M(\Omega,\rho)$ converges to u a.e. and bounded in $L_M(\Omega,\rho)$, therefore $u\in L_M(\Omega,\rho)$ and $u_n\to u$ for the topology $\sigma(L_M(\Omega,\rho),E_{\overline{M}}(\Omega,\rho))$.*

Let Ω an open bounded locally-border lipschitzian in $\mathbb{R}^N$, ρ the weight function and M an N-function. Let the following integrability assumptions :
There exists a real $s > 0$ such that:

$$(M(t))^{\frac{s}{s+1}} \text{ be N-function and that } \rho^{-s} \in L^1(\Omega), \tag{2.6}$$

$$\int_1^{\infty} \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}} dM(t) = \infty, \tag{2.7}$$

$$\lim_{t\to\infty} \frac{1}{M^{-1}(t)} \int_0^{t^{\frac{s+1}{s}}} \frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}} du = 0. \tag{2.8}$$

Remark 2.1 *In the particular case where $M(t) = \frac{|t|^p}{p}(1 < p < \infty)$, the first part of (2.6) is satisfied if $s > \frac{1}{p-1}$.*

Theorem 2.1 *(See[2] Theorem 9-5). Let Ω an open bounded subset of $\mathbb{R}^N$ with locally- lipschitzian and M an N-function. Suppose that assumptions (2.6) - (2.8) are satisfied. The following compact injection is introduced :*

$$W^1L_M(\Omega, \rho) \hookrightarrow\hookrightarrow E_M(\Omega).$$

This study also use the following technical lemmas.

2.5 Some technical lemmas

Lemma 2.3 *Let $f_n, f \in (\Omega)$ such that $f_n \geq 0$ a.e in Ω and $\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$. Consequently $f_n \to f$ strongly in $L^1(\Omega)$.*

Lemma 2.4 *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^1L_M(\Omega, \rho)$, the $F(u) \in W_0^1L_M(\Omega, \rho)$. Additionally, if the set D of discontinuity points of F' is finite, therefore*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e \quad in \ \{x \in \Omega; u(x) \notin D\}, \\ 0 \quad in \ a.e & \{x \in \Omega; u(x) \in D\}. \end{cases}$$

Proof. It is hypothesized that F is also C^1, there exist a sequence $u_n \in D(\Omega)$ such that $u_n \to u$ (mod)$W^1L_M(\Omega, \rho)$. Passing to subsequence, it is assumed that

$$D^\alpha u_n \to D^\alpha u \forall |\alpha| \leq 1 \text{ a.e in } \Omega.$$

From the relation $|F(s)| \leq k|s|$, where k denote the Lipschitz constant for F, and $\frac{\partial}{\partial x_i}F(u_n) = F'(u_n)\frac{\partial u_n}{\partial x_i}$, the results suggest that $F(u_n)$ remains bounded in $W_0^1L_M(\Omega, \rho)$. Thus going to a further subsequence, it was obtained

$$F(u)_n \to w \in W_0^1L_M(\Omega, \rho) \text{ for } \sigma(\Pi L_{M,\rho}, \Pi E_{\overline{M},\rho}),$$

and also by a local application of the compact imbedding theorem, $F(u_n) \to w$ a.e in Ω. Consequently $w = F(u)$, and $f(u) \in W_0^1L_M(\Omega, \rho)$. Finally, by the usual chain rule for weak derivatives,

$$\frac{\partial}{\partial x_i}F(u) = F'(u)\frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega. \tag{2.9}$$

For the general case. Taking convolution with the mollifiers, a sequence is obtained $F_n \in C^\infty(\mathbb{R})$ such that $F_n \to F$ uniformly on each compact, $F_n(0) = 0$ and $|F_n'| \leq k$. For each n, $F_n(u) \in W_0^1L_M(\Omega, \rho)$, and hence (2.9) with F replaced by F_n. Finally (2.9) follows from the generalized chain rule for weak derivatives. The following lemmas follow from the previous lemma.

Lemma 2.5 *Let $u, v \in W_0^1 L_M(\Omega, \rho)$ and let $\omega = min(u, v)$, consequently $\omega \in W_0^1 L_M(\Omega, \rho)$ and*

$$\frac{\partial \omega}{\partial x_i} = \{ \ \frac{\partial u}{\partial x_i} \ a.e \ in \ \{x \in \Omega; u(x) \leq v(x)\}, \ \frac{\partial v}{\partial x_i} \ a.e \ in \ \{x \in \Omega; u(x) > v(x)\}.$$

Proof. Note that min $(u, v) = u - (u - v)^+$ and apply lemma 2.4 with $F(s) = s^+$. The truncate operator is introduced. For a given constant $k > 0$, the function was defined $T_k : I\!R \to I\!R$ as

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \geq k, \\ k\frac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

Lemma 2.6 *Assume that (2.6) - (2.8) holds. Let $u \in W_0^1 L_M(\Omega, \rho)$, and let $T_k(u), k \in R^+$, be the usual truncation the $T_k(u) \in W_0^1 L_M(\Omega, \rho)$. Furthermore, given that the*

$$T_k(u) \to \ markedly \ in \ W_0^1 L_M(\Omega, \rho).$$

Lemma 2.7 *Let u_n be a sequence in $W_0^1 L_M(\Omega, \rho)$ such that $u_n \to\to u$ for the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$. thus $T_k(u_n) \rightharpoonup T_k(u)$ for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$.*

Proof. Since $u_n \rightharpoonup u$ and $W_0^1 L_M(\Omega, \rho) \hookrightarrow\hookrightarrow E_M(\Omega)$ it has found that $u_n \to u$ significantly in $E_M(\Omega)$ and a.e. in Ω, then $T_k(u_n) \to T_k(u)$ a.e. in Ω. On the other hand, for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{|T_k(u_n)|}{\lambda}\right) dx \leq \int_\Omega M\left(\frac{|u_n|}{\lambda}\right) dx,$$

and

$$\int_\Omega M\left(\frac{|\nabla T_k(u_n)|}{\lambda}\right) \rho(x) dx = \int_\Omega M\left(\frac{|T_k'(u_n)||\nabla u_n|}{\lambda}\right) \rho(x) dx \leq \int_\Omega M\left(\frac{|\nabla u_n|}{\lambda}\right) \rho(x) dx,$$

imply that

$$\|T_k(u_n)\|_{1,M,\rho} \leq \|u_n\|_{1,M,\rho}.$$

Subsequently $(T_k(u))_n$ is bounded in $W_0^1 L_M(\Omega, \rho)$ hence by lemma 2.2 , given that the $T_k(u)_n \rightharpoonup T_k(u)$ in $W_0^1 L_M(\Omega, \rho)$
for $\sigma(L_M(\Omega, \rho), E_{\overline{M}}(\Omega, \rho))$.

Lemma 2.8 *If the sequence $u_n \in E_M(\Omega, \rho)$ converges a.e. in Ω with $\rho \in L^1(\Omega)$, it converges in norm in $E_M(\Omega, \rho)$ if and only if the norms are uniformly absolutely continuous, i.e. for each $\epsilon > 0$ there exist $\delta > 0$ such that $\|u_n X_E\|_{M,\rho} < \epsilon$, for all n and $E \subset \Omega$ with $|E| < \delta$.*

Proof. By the same argument introduced in in the proof of lemma 11.2 in [9] it is found that $E_{n,m} = \{x \in \Omega : |u_n(x) - u_m(x)|\}$ where $\alpha = M^{-1}(\frac{\epsilon}{3\|\rho\|_1})$ and with $\delta > 0$ such that

$$\|u_n X_E\|_{M,\rho} < \frac{\epsilon}{3}.$$

It is denoted by $H(E_M(\Omega), r)$ the set of functions u $\in L_M(\Omega)$ whose distance to $E_M(\Omega)$ (with respect to the Orlicz norm) is strictly less than r and by $B_{L_M}(\Omega)(0, r)$ the ball in $L_M(\Omega)$ (with respect to the Orlicz norm) of radius r and center 0.

Lemma 2.9 *Let (Ω) be bounded subset of $I\!R^N$ with finite measure. Let M, R and Q be N-functions such that $Q << R$, and let f be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in I\!R$:*

$$|f(x, s)| \leq b(x) + k_1 R^{-1})(\rho_{-1}(x) Q(k_2|s|)), \tag{2.10}$$

where $0 \leq b(x) \in E_M(\Omega, \rho), \rho \in L^1$ and $k_1, k_2 \in I\!R^+$. The Nemytskii operator

$$N_f(u)(x) = f(x, u(x)),$$

(1)Sends $(H(E_Q(\Omega),\frac{1}{k_2}))^r$ into $L_R(\Omega,\rho)$ and is continuous form $(H(E_Q(\Omega),\frac{1}{k_2}))^r$ to the norm topology of $(LQ(\Omega))^p$ into $L_R(\Omega,\rho)$ to the modular convergence;
(2) Its uniformly bounded on $(B_{L_Q(\Omega)}(0,\frac{1}{k_2}))^p$;
(3) If $b(x)\in L_{R_1}(\Omega,\rho)$ with $R_1 << R, N_f$ is continuous to the norm topology of $E_{R_1}(\Omega,\rho)$.

Proof.(1)Let $u=(u_1,u_2,\cdots,u_n)\in(H(E_Q(\Omega),\frac{1}{k_2}))^p$. Since $d(u_iE_Q(\Omega))<\frac{1}{K_2}(1\leq I\leq P)$, it is confirmed that $\int_\Omega Q(k_2|u|)dx\leq 1$, (see Theorem 10.1[9]). Let $\lambda\geq 2k_1$ such that $\frac{2b(x)}{\lambda}\in k_R(\Omega,\rho)$. By the growth condition (2.10) and convexity of R, it is obtained

$$\int_\Omega R\left(\frac{|f(x,s)|}{\lambda}\right)\rho(x)dx\leq\frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx+\frac{1}{2}\int_\Omega R((k_2|u(x)|))dx<\infty.$$

On the other hand, suppose that

$$u_n\to u\in\left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p,$$

and let $\alpha>0$ such that $d(k_2|u|,E_Q(\Omega))<\alpha<1$ and $d(k_2|u|,E_Q(\Omega))<1-\alpha<1$. It is confirmed that $\frac{k_2}{\alpha}|u|\in K_Q(\Omega)$ and $\frac{k_2}{1-\alpha}|u|\in K_Q(\Omega)$ (see theorem 10.1 [9])and for $\lambda>4k_1$ such that $\frac{4b(x)}{\lambda}\in K_R(\Omega,\rho)$, is observed

$$\int_\Omega R\left(\frac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx$$

$$\leq\int_\Omega R\left(\frac{2b(x)+k_1R^{-1}(\rho^{-1}(x)Q(K_2|u_n|))+k_1R^{-1}(\rho^{-1}(x)Q(K_2|u|))}{\lambda}\right)\rho(x)dx$$
$$\leq\frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)\rho(x)dx+\frac{1-\alpha}{4}\int_\Omega Q\left(\frac{k_2}{1-\alpha}|u_n-u|\right)\rho(x)dx$$
$$+\frac{\alpha}{4}\int_\Omega Q\left(\frac{k_2}{\alpha}|u|\right)dx+\frac{1}{4}\int_\Omega Q(k_2|u|)dx.$$

Since $Q(k_2|u|)\leq Q\left(\frac{k_2}{\alpha}|u|\right)$, the last inequality becomes

$$\int_\Omega R\left(\frac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx$$
$$\leq\frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)\rho(x)dx+\int_\Omega Q\left(\frac{k_2}{1-\alpha}|u_n-u|\right)dx+\int_\Omega Q(k_2|u|)dx,$$

which implies by using the Vitali's theorem

$$f(x,u_n)\to f(x,u)(\text{mod})\text{in } L_R(\Omega,\rho),\ (\text{See [3] p52})$$

for a subsequence denoted again u_n (which holds for the whole sequence).
(2)Let now $u\in(B_{L_Q(\Omega)}(0,\frac{1}{k}))^p$ and let $\lambda\geq 2k$ such that $\int_\Omega R(\frac{2b(x)}{\lambda})\rho(x)dx\leq 1$. By the growth condition (2.10) and the convexity of R, is obtained

$$\int_\Omega R\left(\frac{|f(x,u)|}{\lambda}\right)\rho(x)dx\leq\frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx+\frac{1}{2}\int_\Omega Q(k_2|u(x)|)dx\leq 1,$$

which implies (2).
(3)Suppose that $b(x)\in E_{R_1}(\Omega,\rho)$with$R_1\prec\prec R$. As in(1), since $L_R(\Omega,\rho)\subset E_{R_1}(\Omega,\rho)$, it can be demonstrated that $f(x,u)\in E_{R_1}(\Omega,\rho)$ for all $u\in\left(E_Q(\Omega),\frac{1}{k_2}\right)^p$. Suppose now that

$$u_n\to u\in\left(E_Q(\Omega),\frac{1}{k_2}\right)^p\ \text{in}\ (L_Q(\Omega))^P.$$

$$f(x,u_n) \to f(x,u)(\text{mod}) \text{ in } E_{R_1}(\Omega,\rho).$$

Fix $\epsilon > 0$, as stated above

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \le \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\epsilon}\right)\rho(x)dx$$

$$+\frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\epsilon}R^{-1}(\rho^{-1}(x)Q(k_2|u_n|)\right)\rho(x)dx + \frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\epsilon}R^{-1}(\rho^{-1}(x)Q(k_2|u|)\right)\rho(x)dx.$$

Since $R_1 \prec\prec R$, there exists K' such that $R_1\left(\frac{4k_1}{\epsilon}t\right) \le R(t) + K'$ for all $t \ge 0$. The last inequality can be written as the form

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \le \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\epsilon}\right)\rho(x)dx + \frac{1}{4}\int_\Omega Q(k_2|u_n|)dx$$

$$+\frac{1}{4}\int_\Omega Q(k_2|u|)dx + \frac{k'}{2}\int_\Omega \rho(x)dx.$$

As in (1) by using the Vitali's theorem, can be obtained

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \to 0 \text{ as } n \to \infty,$$

for a subsequence (which holds for the whole sequence). Since ϵ is arbitrary, it is concluded that.

3 Main results

Let Y be a closed subspace of $W^1L_M(\Omega,\rho)$ for $\sigma(\Pi L_{M,\rho}, \Pi E_{\overline{M},\rho})$ and let

$$Y_0 = Y \cap W^1L_M(\Omega,\rho),$$

such that Y is the closure of Y_0 for $(\Pi L_{M,\rho}, \Pi E_{\overline{M},\rho})$. In the next, the complementary system is considered (Y, Y_0, Z, Z_0) generated by Y i.e. Y_0^* can be identified to Z and Z_0^* can be identified to Y by the means $\langle .,. \rangle$. Let the mapping T (associated to the operator A) defined from
$D(T) = u \in Y, a_0(x,u,\nabla u) \in L_{\overline{M}}(\Omega), a_i(x,u,\nabla u) \in L_{\overline{M}}(\Omega)$ into Z by the formula

$$a(u,v) = \int_\Omega a_0(x,u,\nabla u)v(x)dx + \sum_{1\le i\le N}\int_\Omega a_i(x,u,\nabla u)\frac{\partial v(x)}{\partial x_i}\rho(x)dx \quad \forall v \in Y_0.$$

The complementary system is taken into consideration in this study

$$(Y, Y_0, Z, Z_0) = (W_0^1L_M(\Omega,\rho)), (W_0^1E_M(\Omega,\rho)), W^{-1}E_{\overline{M}}(\Omega,\rho), W^{-1}L_{\overline{M}}(\Omega,\rho).$$

Our main results are collected in the following theorem.
Theorem 3.1. Under the assumptions (1.2) - (1.7), (2.6) - (2.8) and $\rho(x)$ be a weight function on Ω satisfy (2.3) there exist an entropy solution u of the problem (1.1).

3.1 Main Lemma.

Lemma 3.1. Let u be a measurable function such that $T_k(u)$ belongs to $W_0^1L_M(\Omega,\rho)$ for every $k > 0$. Then

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)dx + \int_\Omega g(x,u,\nabla u)T_k(u-\phi)dx \le \int_\Omega fT_k(u-\phi)dx, \tag{3.1}$$

1 Introduction

In this study deals with existence of entropy solutions to the following nonlinear Dirichlet problem:

$$\begin{cases} A(u) + g(x, u, \nabla u) = f & \text{in} \quad \Omega, \\ u = \ 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $A(u) = -div(\rho(x)a(x, u, \nabla u)), \Omega$ is a bounded domain of $I\!R(N \geq 2).a(x, u, \nabla u) = (a_i(x, u, \nabla u))_{1\leq i\leq N}$, $a_i : \Omega \times I\!R \times I\!R^N \longrightarrow I\!R$ is a Carathéodory functions (that is measurable with respect to x in Ω for every $(s, \xi) \in I\!R \times I\!R^N$, and continuous with respect to$(s, \xi) \in I\!R \times I\!R^N$ for almost every$x \in \Omega$) such that for all $\xi, \acute{\xi}$ in $I\!R^N$

$$|(a_i(x, s, \xi)| \leq \phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|s|)) + K_i\overline{M}^{-1}M(c_1|\xi|), \tag{1.2}$$

$$(a(x, s, \xi) - a(x, s, \acute{\xi}))(\xi - \acute{\xi}) \geq 0, \tag{1.3}$$

$$a(x, s, \xi)\xi \geq M(\lambda_1|\xi|), \tag{1.4}$$

where c_1, c_2, λ_1 and K_i belongs to $I\!R_+.M, P$ be two N-functions such that $P \prec\prec M$. Furthermore $\overline{M}, \overline{P}$ be the complementary functions of M and P respectively, ρ be a weight function on Ω (that is, measurable and positive a.e. on Ω) and $\phi_i \in E_{\overline{M}}(\Omega, \rho)$.

In addition, the function $g(x, s, \xi)$ is a Carathéodory function satisfying:

$$g(x, s, \xi)s \geq 0, \tag{1.5}$$

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M(|\xi|)), \tag{1.6}$$

$$f \in L^1(\Omega). \tag{1.7}$$

The notion of entropy solutions, used in [11], allows us to give a meaning to a possible solution of (1.1). In fact, in [11], Boccardo proved, for $\rho(x) = 1$ and p such that $2 - 1/N < p < N$, the existence and the regularity of an entropy solution u of problem (1.1). For the case that $\rho(x)$=1, $\phi = 0$ and f is a bounded measure, Bénilan et al. [17] proved the existence and the uniqueness of entropy solutions. The same problem is treated by using the notion of entropy solutions introduced in [24], where $\rho(x) = 1$, $f \in L^1(\Omega)$, and $f \in L^{p'}(\Omega)^N$.
The reason for this, is that $a(x, s, \nabla u)$ does not need to satisfy the strict monotonicity condition that is,

$$\left(a(x, s, \xi) - a(x, s, \acute{\xi})\right)(\xi - \acute{\xi}) > 0, \text{ for all } \ \xi, \acute{\xi} \in I\!R^N, (\xi \neq \acute{\xi}),$$

of a typical Leray-Lions operator but only a large monotonicity that is

$$\left(a(x, s, \xi) - a(x, s, \acute{\xi})\right)(\xi - \acute{\xi}) \geq 0, \text{ for all } \ \xi, \acute{\xi} \in I\!R^N,$$

and $g(x, s, \xi)$ presents the nonlinearity of the problem (1.1).
Bensoussan, Boccardo and Murat [16] proved the existence of solutions for the Dirichlet problem of the form (1.1),where $g(x, s, \xi)$ satisfies :

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + |\xi|^p \quad \text{(natural growth condition)} \ ,$$

$$g(x, s, \xi).s \geq 0 \qquad \text{(sign condition)} \ .$$

Further reference is made to [3, 4] for more details. Benkirane and Elmahi [15] have proved the existence theorem of the problem (1.1) in Orlicz-Sobolev space $W^1L_M(\Omega)$, by assuming a sign condition and a natural growth condition on $g(x, s, \xi)$ of the form:

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M\left(\frac{|\xi|}{\lambda}\right)) \ \ (M(.) \text{ is an N-function }).$$

is equivalent to

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\rho dx + \int_\Omega g(x,u,\nabla u)T_k(u-\phi)dx = \int_\Omega fT_k(u-\phi)dx, \tag{3.2}$$

for every $\phi \in W_0^1L_M(\Omega,\rho)$, and for every $k>0$.
Proof. In fact (3.2) implies (3.1)is easily proved adding and subtracting

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\rho dx + \int_\Omega g(x,u,\nabla u)T_k(u-\phi)\rho dx,$$

and using assumption (1.3). Thus, it remains to prove that (3.1) implies (3.2). Let h and k be positive real numbers, let$\lambda \in]-1,1[$ and $\Psi \in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$.
Choose $\phi = T_h(u-\lambda T_k(u-\Psi)) \in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$ as test function in (3.1), one obtains:

$$I_{hk} \le J_{hk}, \tag{3.3}$$

with

$$I_{hk} = \int_\Omega a(x,u,\nabla T_h(u-\Psi))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx + \int_\Omega g(x,u,\nabla T_h(u-\Psi))T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx,$$

and

$$J_{hk} = \int_\Omega fT_k(u-T_h(u-\lambda T_k(u-\Psi)))dx.$$

Put

$$A_{hk} = \{x\in\Omega, |u-T_h(u-\lambda T_k(u-\Psi))| \le k\},$$

and

$$B_{hk} = \{x\in\Omega, |u-\lambda T_k(u-\Psi)| \le h\},$$

it is derived that

$$\begin{aligned} I_{hk} = &\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx \\ &+\int_{A_{kh}\cap B_{hk}} g(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx \\ &+\int_{A_{kh}\cap B_{hk}^c} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx \\ &+\int_{A_{kh}\cap B_{hk}^c} g(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx \\ &+\int_{A_{kh}^c} a(x,u,\nabla T_h(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx \\ &+\int_{A_{kh}^c} g(x,u,\nabla T_h(u-\Psi)))T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx. \end{aligned}$$

Since $\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))$ is different to zero only on A_{kh}, it follows that

$$\begin{aligned} &\int_{A_{kh}^c} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx \\ &+\int_{A_{kh}^c} g(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx = 0. \end{aligned} \tag{3.4}$$

Furthermore, if $x\in B_{hk}^C$ the result is $\nabla T_h(u-\lambda T_k(u-\Psi)) = 0$ and using (1.4), this implies that,

$$\int_{A_{kh}\cap B^c_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx$$
$$+\int_{A_{kh}\cap B^c_{hk}} g(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx$$
$$=\int_{A_{kh}\cap B^c_{hk}} a(x,u,0)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx$$
$$+\int_{A_{kh}\cap B^c_{hk}} g(x,u,0)T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx=0. \tag{3.5}$$

From (3.4) and (3.5), this yields

$$I_{hk}=\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx$$
$$+\int_{A_{kh}\cap B_{hk}} g(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx.$$

Letting $h\to\infty, |\lambda|\le 1$, it is obtained that

$$A_{kh}\to\{x,|\lambda||T_k(u-\Psi)|\le h\}=\Omega, \tag{3.6}$$
$$B_{hk}\to\Omega \text{ which implies } A_{kh}\cap B_{hk}\to\Omega. \tag{3.7}$$

Which and using Lebesgue theorem, it is concluded that

$$\lim_{h\to+\infty}\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx$$
$$+\lim_{h\to+\infty}\int_{A_{kh}\cap B_{hk}} g(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))T_k(u-T_h(u-\lambda T_k(u-\Psi)))dx$$
$$=\lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)dx$$
$$+\lambda\int_\Omega g(x,u,\nabla(u-\lambda T_k(u-\Psi)))T_k(u-\Psi)dx. \tag{3.8}$$

Thus implies that,

$$\lim_{h\to+\infty} I_{hk}=\lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)dx+\lambda\int_\Omega g(x,u,\nabla(u-\lambda T_k(u-\Psi)))T_k(u-\Psi)dx. \tag{3.9}$$

On the other, it follows that,

$$J_{hk}=\int_\Omega fT_k(u-T_h(u-\lambda T_k(u-\Psi)))dx.$$
$$\lim_{h\to+\infty}\int_\Omega fT_k(u-T_h(u-\lambda T_k(u-\Psi)))dx=\lambda\int_\Omega fT_k(u-\Psi)dx, \tag{3.10}$$

i.e.,

$$\lim_{h\to+\infty} J_{hk}=\lambda\int_\Omega fT_k(u-\Psi)))dx. \tag{3.11}$$

Together (3.9),(3.11) and passing to the limit in (3.3), it follows that,

$$\lambda(\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi))\nabla T_k(u-\Psi)dx)+\lambda(\int_\Omega g(x,u,\nabla(u-\lambda T_k(u-\Psi))T_k(u-\Psi)dx)\le\lambda(\int_\Omega fT_k(u-\Psi)dx),$$

for every $\Psi \in W_0^1 L_M(\Omega,\rho) \cap L^\infty(\Omega)$, and for every $k > 0$. Choosing $\lambda > 0$ dividing by λ , and letting λ and to zero, it is hypothesized that

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)dx + \int_\Omega g(x,u,\nabla u)T_k(u-\Psi)dx \le \int_\Omega fT_k(u-\Psi)dx. \tag{3.12}$$

For $\lambda < 0$ dividing by λ, and letting λ and to zero, hence, the following holds

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)dx + \int_\Omega g(x,u,\nabla u)T_k(u-\Psi)dx \ge \int_\Omega fT_k(u-\Psi)dx. \tag{3.13}$$

From the combination of(3.12)and (3.13), the following equality is deduced:

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)dx + \int_\Omega g(x,u,\nabla u)T_k(u-\Psi)dx = \int_\Omega fT_k(u-\Psi)dx. \tag{3.14}$$

This completes the proof of lemma 3.1.

Definition 3.1 *A measurable function u is called an entropy solution of the highly nonlinear problem (1.1) if*

$$T_k(u) \in W_0^1 L_M(\Omega,\rho), g(x,u,\nabla u) \in L^1(\Omega,\rho),$$

$$\int_\Omega a(x,u,\nabla u).\nabla T_k(u-\phi)dx + \int_\Omega g(x,u,\nabla u).T_k(u-\phi)dx = \int_\Omega fT_k(u-\phi)dx,$$

for any $\phi \in W_0^1 L_\phi(\Omega,\rho) \cap L^\infty(\Omega,\rho)$.

3.2 proof of Theorem 3.1.

3.2.1. Approximate Problem and A priori Estimate.
For $n \in I\!N$, define$f_n := T_n(f)$, and $|g_n(x,u_n,\nabla u_n)| \le |g(x,u,\nabla u)|$. Let u_n be solution in $W_0^1 L_\varphi(\Omega)$ of the problem

$$\begin{cases} \text{- div } (\rho(x)a(x,u_n,\nabla u_n) + g_n(x,u_n,\nabla u_n) = f_n & \text{in} \quad \Omega, \\ u_n = \ 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.15}$$

which exists thanks to [23]. Choosing $T_k(u_n)$ as test function in (3.15), it is obtained that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)dx + \int_\Omega g(x,u_n,\nabla u_n).T_k(u_n)dx = \int_\Omega f_n T_k(u_n)dx,$$

using $\nabla T_k(u_n) = \nabla u_{nX\{|u_n|\le k\}}$ and thanks to assumption (1.4), (1.7), it is observed the following expression

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)dx + \int_\Omega g(x,u_n,\nabla u_n).T_k(u_n)dx \ge \int_\Omega M(\lambda_1|\nabla T_k(u_n)|)dx,$$

$$\int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \le k\|f\|_{L^1}(\Omega). \tag{3.16}$$

$$\int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \le C_1 k, \tag{3.17}$$

where C_1 is a constant independently of n.
3.2.2. Locally Convergence of u_n in Measure.
Taking $\frac{1}{\lambda}|T_k(u_n)|$ in (3.15) and using(3.17), one has

$$\int_\Omega \rho(x)M(\lambda_1\frac{|\nabla T_k(u_n)|}{\lambda})dx \le \int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \le C_1 k. \tag{3.18}$$

$$meas\{|u_n > k|\} \leq \frac{1}{\inf_k M(\frac{K}{\lambda})} \int_{\{} |u_n > k|\} M \frac{|u_n|(x)}{\lambda} dx$$
$$\leq \frac{1}{\inf_k M(\frac{K}{\lambda})} \int_\Omega M(\frac{1}{\lambda}|T_k(u_n)|)dx$$
$$\leq \frac{C_1 k}{\inf_k M(\frac{K}{\lambda})} \quad \forall n, \forall k \geq 0. \tag{3.19}$$

For any $\beta > 0$, it follows that

$$meas\{|u_n - u_m| > \beta\} \leq meas\{|u_n| > k\} + meas\{|u_m| > k\} + meas\{|T_K(u_n)| - T_K(u_m) > \beta\},$$

and so that

$$meas\{|u_n - u_m| > \beta\} \leq \frac{2C_1 k}{\inf_k M(\frac{K}{\lambda})} + meas\{|T_K(u_n)| - T_K(u_m) > \beta\}. \tag{3.20}$$

By using (3.17) and Poincaré inequality in weighted Orlicz-Sobolev spaces, it is deduced that $(T_k(u_n))$ is bounded in $W_0^1 L_M(\Omega, \rho)$, and there exists $\omega_k \in W_0^1 L_M(\Omega, \rho)$ such that $T_k(u_n) \rightharpoonup \omega_k$ weakly in $W_0^1 L_M(\Omega, \rho)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$; significantly in $E_{\overline{M}}(\Omega, \rho)$ and a.e. in Ω.

Consequently, it can be assumed that$(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω. Let $\epsilon > 0$, by (3.20) and the fact that $\frac{2C_1 k}{\inf_k M(\frac{K}{\lambda})} \to 0$ as $k \to +\infty$ there exists some $k = k(\epsilon) > 0$ such that

$$meas\{|u_n - u_m| > \lambda\} < \epsilon, \text{ for all } n, m \geq h_0(k(\epsilon), \lambda).$$

This proves that u_n is a Cauchy sequence in measure, thus, u_n converges almost everywhere to some measurable function u. Finally, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in W_0^1 L_M(\Omega, \rho)$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega, \rho) \text{ for } \sigma(\Pi L_{M,\rho}, \Pi E_{\overline{M},\rho}) \\ \quad u_n \to u \text{ strongly in } E_M(\Omega, \rho) \text{ and a.e. in } \Omega. \end{cases} \tag{3.21}$$

3.2.3. An Intermediate Inequality.
It is shown in the following step that for $\phi \in W_0^1 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, it follows that

$$\int_\Omega a(x, u_n, \nabla\phi).\nabla T_k(u_n - \phi)dx + g(x, u_n, \nabla\phi).T_k(u_n - \phi) \leq \int_\Omega f_n T_k(u_n - \phi)dx. \tag{3.22}$$

It is now chosen that $T_K(u_n - \phi)$ as test function in (3.15), with ϕ in $W_0^1 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, hence, it follows that

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \phi)dx + g(x, u_n, \nabla\phi).T_k(u_n - \phi) = \int_\Omega f_n T_k(u_n - \phi)dx. \tag{3.23}$$

Adding and subtracting the term $\int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)dx$ i.e.,

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \phi)dx + \int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)dx - \int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)dx$$
$$+ \int_\Omega g(x, u, \nabla u_n).T_k(u_n - \phi)dx = \int_\Omega f_n T_k(u_n - \phi)dx.T_k(u_n - \phi). \tag{3.24}$$

Thanks to assumption (1.3)and the definition of truncation function, it is clear that

$$\int_\Omega a(x, u_n, \nabla u_n) - a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)dx \geq 0. \tag{3.25}$$

Combining(3.23)and (3.25), this yields (3.22).
3.2.4. The Equi-integrability of the Nonlinearities.
It is intended to prove that

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{3.26}$$

In particular, it is enough to prove the equi-integrable of $g_n(x, u_n, \nabla u_n)$. To this purpose. Let $u_n - T_l(u_n - v_0 - T_h(u_n - v_0))$ as test function in (P_n), it follows that

$$\int_{\{|u_n - v_0| \geq h+1\}} |g_n(x, u_n, \nabla u_n)| dx \leq \int_{\{|u_n - v_0| \geq h\}} (|f_n| + \delta(x)) dx.$$

Let $\epsilon > 0$, therefore $h(\epsilon) \geq 1$ such that

$$\int_{\{|u_n - v_0| \geq h(\epsilon)\}} |g_n(x, u_n, \nabla u_n)| dx < \frac{\epsilon}{2}. \tag{3.27}$$

For any measurable subset $E \subset \Omega$, it is obtained that

$$\int_E |g_n(x, u_n, \nabla u_n)| dx \leq \int_E b(h(\epsilon) + \|v_0\|_\infty) \left(c(x) + M(\nabla T_{h(\epsilon)+\|v_0\|_\infty}(u_n))\right) dx$$
$$+ \int_{\{|u_n - v_0| \geq h(\epsilon)\}} |g(x, u_n, \nabla u_n)| dx.$$

In view of (1.6), hence, it is seen that there exist $\eta(\epsilon) > 0$ such that

$$\int_E b(h(\epsilon) + \|v_0\|_\infty) \left(c(x) + M(\nabla T_{h(\epsilon)+\|v_0\|_\infty}(u_n))\right) dx < \frac{\epsilon}{2}, \text{ for all } E \text{ such that } |E| < \eta(\epsilon). \tag{3.28}$$

Finally, combining (3.27) and (3.28), one easily has $\int_E |g_n(x, u_n, \nabla u_n)| dx < \epsilon$ for all E such that $|E| < \eta(\epsilon)$,
which implies (3.26).
3.2.5. Passing the Limit.
It shall be demonstrated that for $\phi \in W_0^1 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, it can be shown that

$$\int_\Omega a(x, u, \nabla \phi) \nabla T_k(u - \phi) dx + \int_\Omega g(x, u, \nabla u).T_k(u - \phi) dx \leq \int_\Omega f T_k(u - \phi) dx.$$

Firstly, we claim that
$\int_\Omega a(x, u_n, \nabla \phi) \nabla T_k(u_n - \phi) dx + \int_\Omega g(x, u_n, \nabla u_n).T_k(u_n - \phi) dx$
$\to \int_\Omega a(x, u, \nabla \phi) \nabla.T_k(u - \phi) dx + \int_\Omega g(x, u, \nabla \phi).T_k(u - \phi) dx$ as $n \to +\infty$. Since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^1 L_M(\Omega, \rho)$, with $M = k + \|\phi\|_\infty$,

$$T_k(u_n - \phi) \rightharpoonup T_k(u - \phi) \text{ in } W_0^1 L_M(\Omega, \rho), \tag{3.29}$$

Show that

$$a(x, T_M(u_n), \nabla \phi) \to a(x, T_M(u), \nabla \phi) \text{ markedly in } (L_{\overline{M}}(\Omega))^N.$$

Thanks to assumption (1.2), it is obtained

$$|a_i(x, T_M(u_n), \nabla \phi)| \leq |(\phi)_i(x)| + k_i \overline{p}^{-1}(\rho^{-1}(x) M(c_2 |T_M(u_n)|)) + k_i \overline{M}^{-1} M(c_1 |\nabla \phi|),$$

with β and μ are positive constants. Since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^1 L_M(\Omega, \rho)$ and $W_0^1 L_M(\Omega, \rho) \hookrightarrow\hookrightarrow L_{\overline{M}}(\Omega, \rho)$, consequently $T_M(u_n) \rightharpoonup T_M(u)$ highly in $L_M(\Omega, \rho)$ and a.e. in Ω, hence

$$|a(x, T_M(u_n), \nabla \phi)| \to |a(x, T_M(u), \nabla \phi)| \text{ a.e. in } \Omega,$$

and

$$|\phi_i(x)| + k_i\overline{p}^{-1}(\rho^{-1}(x)M(c_2|T_M(u_n)|)) + k_i\overline{M}^{-1}M(c_1|\nabla\phi|) \to$$
$$|\phi_i(x)| + k_i\overline{p}^{-1}(\rho^{-1}(x)M(c_2|T_M(u)|)) + k_i\overline{M}^{-1}M(c_1|\nabla\phi|),$$

a.e. in Ω. Depending on Vitali's theorem, it is deduced that

$$\int_\Omega a(x, T_M(u_n), \nabla\phi) \to \int_\Omega a(x, T_M(u), \nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega, \rho))^N, \text{ as } n \to \infty. \tag{3.30}$$

Hence, it is observed that (3.30)

$$\int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)dx \to \int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)dx \text{ as } n \to +\infty. \tag{3.31}$$

To pass to the limit in the approximate problem, it is shown that

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ in } L^1(\Omega, \rho). \tag{3.32}$$

Secondly, it is shown that

$$\int_\Omega f_n T_k(u_n - \phi)dx \to \int_\Omega f T_k(u - \phi)dx. \tag{3.33}$$

It is found that $f_n T_k(u_n - \phi) \to f T_k(u - \phi)$ a.e. in Ω and $|f T_k(u_n - \phi)| \leq k|f|$, using Vitali's theorem, (3.32) is obtained. Thanks to (3.25) and (3.32) allow to pass to the limit in the inequality (3.22), so that $\forall\phi \in W_0^1 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, it follows that

$$\int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)dx + \int_\Omega g(x, u, \nabla u).T_k(u - \phi)dx \leq \int_\Omega f T_k(u - \phi)dx.$$

In view of main lemma, from the above, hence, it is concluded that u is an entropy solution of the problem (1.1). This completes the proof of theorem 3.1.

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